

A Metastability Result for the Contact Process on a Random Regular Graph

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Abstract

In this paper we study the metastability of the contact process on a random regular graph. We show that the extinction time of the contact process, when initialized so that all vertices are infected at time 0, grows exponentially with the vertex number. Moreover, we show that the extinction time divided by its mean converges to a unit exponential distribution in law.

1 Introduction

The contact process $(\xi_t)_{t \geq 0}$ with infection parameter λ on a connected, locally finite graph $G = (\mathcal{V}_G, \mathcal{E}_G)$ is a continuous-time Markov chain that evolves as follows. For each t the random variable ξ_t takes value in $\{\text{subsets of } \mathcal{V}_G\}$; we regard the elements of ξ_t as infected vertices. Each infected vertex recovers with rate 1; each healthy vertex (a vertex in $\mathcal{V}_G \setminus \xi_t$) becomes infected at rate λ times the number of infected neighbors.

The contact process on a finite graph G will eventually reach the absorbing state \emptyset . Of natural interest is the *time to extinction* τ_G (defined to be the time of the first visit to \emptyset) when the contact process is started from the full-occupancy state. Since large connected graphs G will contain long linear chains, when the infection parameter λ exceeds the critical value λ_c for the contact process on \mathbb{Z} the contact process on G can be expected to survive for a long time, eventually reaching a quasi-stationary state that persists until, by chance, a large number of infected vertices almost simultaneously become healthy, leading to subsequent extinction. This phenomenon has become known generically as “metastability”. Metastability for contact process was first introduced in [3] for the one-dimensional finite cube. Subsequently, there have been various studies of metastability on other class of graphs, for example, linear chains in [18], the d -dimensional finite cube in [13], power law random graphs in [15], finite trees in [4], and a family of finite graphs in [14].

In [7] the contact process on random d -regular graphs ($d \geq 3$) is studied. A random regular graph $G \sim \mathcal{G}(n, d)$ is a graph chosen uniformly from all d -regular graphs with n vertices. Such a graph locally looks like a tree, but globally it differs significantly from a

finite tree. Nevertheless, certain techniques and results that have been developed for studying the contact processes on trees will be of use here. In [8, 17, 19] it is shown that when $d \geq 3$, there exist constants $0 < \lambda_1(\mathbb{T}^d) < \lambda_2(\mathbb{T}^d) < \infty$ that demarcate different phases for the contact process on an infinite d -regular tree \mathbb{T}^d . If $\lambda < \lambda_1(\mathbb{T}^d)$, the contact process started from a finite initial configuration will almost surely die out; when $\lambda > \lambda_2(\mathbb{T}^d)$ it has positive probability of local survival; and when λ is in between then it with positive probability survives globally but almost surely dies out locally.

In this paper we investigate the metastable behavior of the contact process on random regular graphs $G \sim \mathcal{G}(n, d)$ with infection parameter $\lambda > \lambda_1(\mathbb{T}^d)$. Denote by $(\xi_t^A)_{t \geq 0}$ the contact process on G with initial configuration $A \subset \mathcal{V}_G$. When $A = \mathcal{V}_G$ we use $(\xi_t)_{t \geq 0}$ as shorthand; if $A = \{u\}$, we write ξ_t^u instead of $\xi_t^{\{u\}}$. Since the underlying graph $G \sim \mathcal{G}(n, d)$ is random, we say that a property holds for *asymptotically almost every* G if the set of graphs in $\mathcal{G}(n, d)$ which satisfy the property has probability tending to 1 as $n \rightarrow \infty$.

Throughout this paper we fix $\lambda > \lambda_1(\mathbb{T}^d)$, and we let $p_\lambda > 0$ be the chance that ζ_t^O , a contact process with initial state $\{O\}$ on \mathbb{T}^d , survives forever. To emphasize conditional probability and expectation given the graph G we use notations \mathbb{P}_G and \mathbb{E}_G . The word “typical” in this paper means asymptotically almost every. Also, all $o(1)$ terms tend to 0 uniformly in n (independent of G).

The main results of this paper are Theorem 1.1 and 1.2.

Theorem 1.1. *There exists $\beta > 0$ such that for asymptotically almost every $G \sim \mathcal{G}(n, d)$,*

$$\mathbb{P}_G\{\xi_{\exp(\beta n)} \neq \emptyset\} = 1 - o(1),$$

where $\{\xi_t\}_{t \geq 0}$ is the contact process with initial configuration \mathcal{V}_G .

Theorem 1.2. *For asymptotically almost every $G \sim \mathcal{G}(n, d)$, the distribution of $\tau_G/\mathbb{E}\tau_G$ converges to an exponential distribution with mean 1, where $\tau_G = \inf\{t \geq 0 : \xi_t = \emptyset\}$ is the extinction time of the contact process started from \mathcal{V}_G .*

These theorems are proved in sections 2 and 3, respectively. While preparing this paper, we learned that J.-C. Mourrat and D. Valesin [16] have independently established Theorem 1.1. Because our proof is somewhat different from theirs, and because Theorem 1.1 is a key complement to Theorem 1.2, we include it in section 2.

2 Exponential extinction time

The goal of this section is to prove Theorem 1.1.

Here are some important facts. Let ζ_t^O be a contact process on \mathbb{T}^d with initial state $\{O\}$. In [11, 12] it is shown that

Theorem 2.1. *There exist constants $c_\lambda, C(d) \in \mathbb{R}$ such that*

$$e^{c_\lambda t} \leq \mathbb{E}|\zeta_t^O| \leq C(d)e^{c_\lambda t}.$$

Moreover, if $\lambda > \lambda_1(\mathbb{T}^d)$ then $c_\lambda > 0$.

Throughout this paper $c_\lambda > 0$ will be the constant in the above theorem.

In [7] (Proposition 5.2 and 5.3) it is shown that

Proposition 2.2. *Fix $0 < \varepsilon < 1/8$. For asymptotically almost every $G \sim \mathcal{G}(n, d)$, there are at least $(1 - o(1))n$ vertices (call them “good” vertices) in G , such that for each good vertex u ,*

$$\mathbb{P}_G\{\xi_{(1+\varepsilon)\log n/c_\lambda}^u \neq \emptyset\} = (1 - o(1))p_\lambda.$$

Proposition 2.3. *Fix $\delta > 0$ and $0 < \varepsilon < 1/8$. Then for asymptotically almost every $G \sim \mathcal{G}(n, d)$, if u is a good vertex in Proposition 2.2, then*

$$\mathbb{P}_G\{(1 - \delta)np_\lambda \leq |\xi_{(1+\varepsilon)\log n/c_\lambda}^u| \leq (1 + \delta)np_\lambda \mid \xi_{(1+\varepsilon)\log n/c_\lambda}^u \neq \emptyset\} = 1 - o(1).$$

Recall that for a graph $G = (\mathcal{V}_G, \mathcal{V}_E)$, the edge expansion parameter is defined as

$$\Psi_E(G, k) = \min_{S \subset \mathcal{V}_G, |S| \leq k} \frac{|E(S, S^c)|}{|S|},$$

where $E(S, S^c) \subset \mathcal{V}_E$ is the set of edges with one vertex in S and the other vertex in S^c . It is shown in [5] (Theorem 4.16) that

Theorem 2.4. *Let $d \geq 3$. Then for every $\delta > 0$ there exists $\varepsilon > 0$ such that for asymptotically almost every $G \sim \mathcal{G}(n, d)$, $\Psi_E(G, \varepsilon n) \geq d - 2 - \delta$.*

Fix an integer $M > 0$. Suppose $U \subset \mathcal{V}_G$ is of size αn , where $\alpha > 0$. We remove every vertex in U whose M -neighborhood is not a tree, and denote the remaining vertex set by U' . We claim $|U'| = \alpha n - o(n)$. Here we are using the following fact shown in [10] (Lemma 3.2):

Proposition 2.5. *For asymptotically almost every $G \sim \mathcal{G}(n, d)$, it has at most $o(n)$ vertices whose $\lfloor \log_{d-1} \log n \rfloor$ -neighborhoods in G are not tree-like.*

Since the cardinality of U' and U are on the same order of magnitude, without loss of generality, let us assume that all vertices in U have tree-like M -neighborhoods in G .

We classify vertices in U into 2 categories by looking at their M -neighborhoods in G in the following way. For $v \in U$, let $B(v, M)$ be the induced subgraph containing all vertices in v 's M -neighborhood in G . $B(v, M) \setminus \{v\}$ has d connected components, call them $C_1(v), C_2(v), \dots, C_d(v)$. If $C_i(v)$ contains no other vertices in U , call it a *free branch of depth M of v* .

- Color v *black* if at least one of $C_1(v), C_2(v), \dots, C_d(v)$ is a free branch of depth M of v .
- Color v *white* if none of $C_1(v), C_2(v), \dots, C_d(v)$ is a free branch of depth M of v .

Proposition 2.6. *Fix $M \in \mathbb{N}$. There exists $\varepsilon = \varepsilon(M) > 0$, such that for asymptotically almost every $G \sim \mathcal{G}(n, d)$ the following statement holds: for any set $U \subset \mathcal{V}_G$ satisfying $|U| \leq \varepsilon n$ and that every vertex in U has its M -neighborhood in G being a tree, then U has at least $|U|/4$ black vertices.*

Proof. We will construct a subset of vertices $W \subset \mathcal{V}_G$. First of all, W contains all vertices in U . Moreover, we are going to add some vertices into W based on the white vertices of U . Let $v \in U$ be a white vertex. In each of $C_1(v), C_2(v), \dots, C_d(v)$ there must be at least another vertex in U . Suppose $x \in U \cap C_i(v)$, then for the pair (v, x) , we add into W every vertex along the (unique) geodesic between v and x . We repeat this operation for every possible pair (v, x) to obtain W .

Such constructed W contains 3 types of vertices: black vertices of U , white vertices of U , and the vertices which are added by the above procedure (color them *grey*). Now let us count their contributions to $E(W, W^c)$.

- A white vertex will contribute 0 edge to $E(W, W^c)$. This is because all of its d neighboring vertices are already in W by our construction.
- A black vertex can contribute at most d edges to $E(W, W^c)$, possibly fewer.
- A grey vertex can contribute at most $d - 2$ edges to $E(W, W^c)$, possibly fewer. This is because by our construction a grey vertex must be sitting on the geodesic between two other vertices in U and therefore at least 2 out of its d neighboring vertices are already in W .

Suppose in U there are w white vertices, b black vertices. Then g , the number of grey vertices in W , satisfies $g \leq (d + d(d - 1) + \dots + d(d - 1)^{M-1})w := N_M w$.

Due to Theorem 2.6, there exists ε_M such that on a typical random regular graph G , $\Psi_E(G, \varepsilon_M n) \geq d - 2 - (3d - 8)/(3N_M + 4)$. This forces the following inequality (provided $b + w + g \leq \varepsilon_M n$),

$$0w + db + (d - 2)g \geq E(W, W^c) \geq (d - 2 - \frac{3d - 8}{3N_M + 4})(w + b + g).$$

Together with $g \leq N_M w$, we conclude that

$$\frac{b}{b + w} \geq \frac{1}{4}.$$

Therefore take $\varepsilon'_M = \varepsilon_M / (N_M + 1)$ (this guarantees that if $b + w \leq \varepsilon'_M n$ then $b + w + g \leq \varepsilon_M n$). As long as $|U| \leq \varepsilon'_M n$ and every vertex in U has its M -neighborhood in G being a tree, then U has at least $|U|/4$ black vertices. \square

We need the following result about the growth rate of the severed contact process on a tree, shown in [7] (Proposition 2.2). A severed contact process $\{\eta_t^O\}_{t \geq 0}$ is a version of contact process on \mathbb{T}^d with initial configuration $\{O\}$ where we do not allow infections to come across $d - 1$ edges connected to O .

Proposition 2.7. *There exists $A = A(\lambda, d) > 0$ such that $\mathbb{E}|\eta_t^O| \geq A \exp(c_\lambda t)$, for all $t \geq 0$.*

Let Δ_M be the following finite graph. Fix the root O in \mathbb{T}^d . We first remove $d - 1$ edges connected to O in \mathbb{T}^d . For the remaining graph, the connected component of the M -neighborhood of O is called Δ_M . In Δ_M , O and vertices at distance M from O are of degree 1; other vertices are of degree d . Δ_M is a tree.

Let $\{\eta_t^{O, \Delta_M}\}_{t \geq 0}$ be a contact process on Δ_M with initial configuration $\{O\}$. The following corollary is easily obtained from Proposition 2.7.

Corollary 2.8. *For every $N > 0$, there exist $T > 0$ and $M \in \mathbb{N}$ such that $\mathbb{E}|\eta_T^{O, \Delta_M}| \geq N$.*

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Let G be a typical graph as in Proposition 2.4, 2.5 and 2.6.

In Corollary 2.8, take $N = 10$ so that there exist T and M satisfying $\mathbb{E}|\eta_T^{O, \Delta_M}| \geq 10$. Without loss of generality assume $T \geq 1$. Furthermore, take $L > 0$ large enough such that $\mathbb{E} \min(|\eta_T^{O, \Delta_M}|, L) \geq 9$. Fix such choices of T , M and L .

Let $\varepsilon = \varepsilon(2M)$ be the constant in Proposition 2.6. As long as $U \subset \mathcal{V}_G$ is of size εn , then there will be at least $\varepsilon n/2$ vertices in U such that each vertex has its $2M$ -neighborhood being a tree, and by Proposition 2.6 there will be at least $\varepsilon n/8$ black vertices. We enumerate the black vertices to be v_1, v_2, \dots, v_k where $k \geq \varepsilon n/8$.

Now for each v_i , it has one free branch of depth $2M$ (if more than one, specify one). Add v_i (and the edge connected to v_i) to its free branch of depth M (as a subgraph of the branch of depth $2M$) and we obtain a subgraph isomorphic to Δ_M . So each v_i is associated with a copy of Δ_M . Here we use Δ_M instead of Δ_{2M} to ensure they are disjoint. For each v_i , we run an independent contact process on its copy of Δ_M , call it $\{\eta_t^{v_i}\}_{t \geq 0}$. In a standard way we can couple $\cup_{i=1}^k \eta_T^{v_i}$ and $\cup_{i=1}^k \xi_T^{v_i}$ together so that $\cup_{i=1}^k \eta_T^{v_i}$ is always dominated by $\cup_{i=1}^k \xi_T^{v_i}$.

Let $X_i = \min(|\eta_T^{v_i}|, L)$, then $\mathbb{E}X_i \geq 9$, and $0 \leq X_i \leq L$. Furthermore $(X_i)_{1 \leq i \leq k}$ are i.i.d. random variables, so by Hoeffding's inequality,

$$\mathbb{P}\left\{\sum_{i=1}^k (X_i - 9) \leq -k\right\} \leq \exp\left(-\frac{2k}{L^2}\right) \leq \exp\left(-\frac{\varepsilon n}{4L^2}\right),$$

in other words, after time T , with probability at least $1 - \exp(-\varepsilon n/(4L^2))$, we will observe at least $8k \geq \varepsilon n$ infections in $\cup_{i=1}^k \eta_T^{v_i}$.

To summarize, as long as we run the contact process with initial configuration whose cardinality is εn , then after time T , with probability more than $1 - \exp(-\varepsilon n/(4L^2))$, the outcome will have cardinality at least εn . Let $\beta = \varepsilon/(8L^2)$, we conclude

$$\mathbb{P}_G\{\xi_{\exp(\beta n)T}^G \neq \emptyset\} \geq 1 - \exp(\beta n) \exp\left(-\frac{\varepsilon n}{4L^2}\right) = 1 - o(1).$$

□

Remark 2.9. *One can slightly change the above proof to show the following statement. There exists a constant $\varepsilon_0 > 0$. For every $0 < \varepsilon \leq \varepsilon_0$, there exists $\beta_\varepsilon > 0$, such that for asymptotically almost every $G \sim \mathcal{G}(n, d)$, for any $U \subset \mathcal{V}_G$ with $|U| = \varepsilon n$, we have*

$$\mathbb{P}_G\{\xi_{\exp(\beta_\varepsilon n)}^U \neq \emptyset\} = 1 - o(1).$$

3 Metastability

Let $\tau_G = \inf\{t \geq 0 : \xi_t = \emptyset\}$ be the extinction time of the contact process started from full occupancy on G . In this section we are devoted to proving Theorem 1.2. The key ingredient is the following proposition in [13] (Proposition 2.1):

Proposition 3.1. *Suppose there is a sequence of graphs $G_n = (\mathcal{V}_n, \mathcal{E}_n)$. Let τ_n be the extinction of the contact process started from full occupancy on G_n . Also, for arbitrary $U \subset \mathcal{V}_n$, we couple ξ_t^U and $\xi_t^{\mathcal{V}_n}$ in the standard way. Suppose there exist two sequences of positive real numbers $a(n)$ and $b(n)$, both tending to infinity, such that as $n \rightarrow \infty$,*

1. $a(n)/b(n) \rightarrow 0$;
2. $\sup_{U \subset \mathcal{V}_n} \mathbb{P}\{\xi_{a(n)}^U \neq \emptyset, \xi_{a(n)}^U \neq \xi_{a(n)}^{\mathcal{V}_n}\} \rightarrow 0$;
3. $\mathbb{P}\{\xi_{b(n)}^{\mathcal{V}_n} \neq \emptyset\} \rightarrow 1$.

Then $\tau_n/\mathbb{E}\tau_n$ converges in distribution to an exponential distribution with mean 1.

Now from Theorem 1.1 we can take $b(n) = \exp(\beta n)$. It suffices to find an appropriate sequence $a(n)$ such that item 1 and 2 in Proposition 3.1 hold.

Lemma 3.2. *There exists $\gamma > 0$, such that for asymptotically almost every $G \sim \mathcal{G}(n, d)$, for any two vertices u, v of G ,*

$$\mathbb{P}_G\{v \in \xi_{2^{\log_{d-1} n}}^u\} \geq n^{-\gamma}.$$

Proof. By [2], the diameter of a typical random regular graph is $(1+o(1)) \log_{d-1} n$. Therefore on such a graph, for any pair of vertices (u, v) , the graph distance between u and v is no more than $(1+o(1)) \log_{d-1} n$. Assume $\text{dist}(u, v) = l$, this means we can find a sequence of vertices $(w_i)_{0 \leq i \leq l}$, with $w_0 = u$, $w_l = v$, and that w_i is connected to w_{i+1} in the graph.

One way of observing $\{v \in \xi_{2^{\log_{d-1} n}}^u\}$ is as follows. In time interval $[i, i+1]$ for $0 \leq i \leq l-1$, we require the infection at vertex w_i to go to w_{i+1} , and stay there alive till the end of the time interval. This will happen with probability $p \geq (1 - e^{-\lambda})e^{-1}$. In time interval $[l, 2 \log_{d-1} n]$, we require the infection at v to stay alive. This will have probability at least $e^{-(2 \log_{d-1} n - l)}$. Therefore, the overall probability is at least $p^l e^{-(2 \log_{d-1} n - l)} \geq (1 - e^{-\lambda})^{2 \log_{d-1} n} e^{-2 \log_{d-1} n} \geq n^{-\gamma}$ where $\gamma = 2/\log(d-1) + 2\log(e^\lambda/(e^\lambda - 1))/\log(d-1)$. \square

Lemma 3.3. *There exists $\delta_0 > 0$, such that for asymptotically almost every $G \sim \mathcal{G}(n, d)$, for any $v \in \mathcal{V}_G$,*

$$\mathbb{P}_G\{|\xi_{n^{2\gamma}}^v| \geq \delta_0 n \mid \xi_{n^{2\gamma}}^v \neq \emptyset\} = 1 - o(n^{-2}).$$

Proof. For asymptotically almost every $G \sim \mathcal{G}(n, d)$, there are $n - o(n)$ good vertices by Proposition 2.2. In particular, there is at least one good vertex. We fix one good vertex in G , call it w . Our idea is as follows. We chop the time interval $[0, n^{2\gamma}]$ into $[0, T], [T, 2T], \dots, [(M-1)T, MT]$, where $M = \lfloor n^{2\gamma}/T \rfloor$ and T to be specified. We will construct an event and see if it happens in each interval $[iT, (i+1)T]$. The chance that it happens in each time interval is on the order of $n^{-\gamma}$. This event is constructed such that as long as in at least one of these intervals the event happens, then with probability approaching 1 at the end we will see $\delta_0 n$ infections.

Here are more details. Let $T = 2 \log_{d-1} n + (1 + \varepsilon) \log n / c_\lambda$, where this $\varepsilon > 0$ is the same as in Proposition 2.2 and 2.3. We say that we observe a success in time interval $[(i-1)T, iT]$ if the following two events happen.

1. $w \in \xi_{(i-1)T + 2 \log_{d-1} n}^v$;

$$2. |\xi_{iT}^v| \geq np_\lambda/2.$$

Conditional on $\xi_{n^{2\gamma}}^v \neq \emptyset$, for sure $\xi_{(i-1)T}^v \neq \emptyset$, so by Lemma 3.2, $\{w \in \xi_{(i-1)T+2\log_{d-1} n}^v\}$ happens with probability at least $n^{-\gamma}$ (notice that the event $\{w \in \xi_{(i-1)T+2\log_{d-1} n}^v\}$ is positively correlated with the event $\{\xi_{n^{2\gamma}}^v \neq \emptyset\}$). Given $\{w \in \xi_{(i-1)T+2\log_{d-1} n}^v\}$, since w is a good vertex, by Proposition 2.3, $|\xi_{iT}^v| \geq np_\lambda/2$ will happen with probability at least $p_\lambda/2$.

Therefore overall we have order $(n^{2\gamma}/\log n)$ trials, each with success probability at least $p_\lambda n^{-\gamma}/2$, so it is easy to conclude that the chance of having at least 1 success is $1 - o(n^{-2})$. Given that we observe a success, which means that we observe $p_\lambda n/2$ infections at some time between $[0, n^{2\gamma}]$, from the proof of Theorem 1.1 and Remark 2.9, we know that the chance of $p_\lambda n/2$ infections not lasting exponentially long time before the size of infections shrinks to $\delta_0 n$ is exponentially small in n , where δ_0 can be taken as $\min(\varepsilon_0, p_\lambda/2)$. Therefore the overall probability is $(1 - o(n^{-2})) \times (1 - o(e^{-\beta n})) = 1 - o(n^{-2})$. \square

We call a sequence of vertices $v_0, \dots, v_L \in \mathcal{V}_G$ a *path* of length L if $v_i v_{i+1} \in \mathcal{E}_G$ for $0 \leq i \leq L-1$. We say such path has endpoints v_0 and v_L . A path of length 0 is allowed, where the endpoints are identical.

Lemma 3.4. *For every $r > 0$, for asymptotically almost every $G \sim \mathcal{G}(n, d)$, there exist $L_r \in \mathbb{N}$ and $c_r > 0$, such that for any $U, W \subset \mathcal{V}_G$ with $|U| = |W| = rn$, there exists $c_r n$ paths in G so that*

- Each path is of length no more than L_r .
- Each path has one endpoint in U and the other endpoint in W .
- None of the paths share the same vertex.

Proof. From [1] and [6], there exists $h_d > 0$ so that for a typical d -random regular graph G its (vertex) isoperimetric constant is at least h_d . Without loss of generality, we assume $r < 1/2$. For $i \in \mathbb{N}$, Let $U_i = \{v \in \mathcal{V}_G : \text{dist}(v, U) \leq i\}$ and $W_i = \{v \in \mathcal{V}_G : \text{dist}(v, W) \leq i\}$. By the definition of the isoperimetric constant, U_1 has cardinality at least $(1 + h_d)rn$, and that U_{i+1} has cardinality at least $(1 + h_d)|U_i|$, provided that $|U_i| \leq n/2$. The same holds for W_i .

Take $L_r = 2\lceil \log_{1+h_d}(1/(2r)) \rceil + 2$, we claim that $|U_{L_r/2}| \geq \frac{d+2h_d}{2d+2h_d}n$. This is because from the way we choose L_r , for sure there will be some $k \leq L_r/2 - 1$ so that k is the smallest integer satisfying $|U_k| \geq n/2$. If $|U_k| \geq \frac{d+2h_d}{2d+2h_d}n$ then we are done. Otherwise, since $|U_k^c| \leq n/2$, the number of vertices in U_k that are adjacent to U_k^c is at least $h_d|U_k^c|$, and that the number of vertices in U_k^c that are adjacent to U_k is at least $h_d|U_k^c|/d$. In this case

$$|U_{k+1}| \geq |U_k| + \frac{h_d}{d}|U_k^c|.$$

Since $|U_k| \geq n/2$ and $|U_k^c| \geq n - \frac{d+2h_d}{2d+h_d}n$, we have

$$|U_{k+1}| \geq \frac{n}{2} + \frac{h_d}{d} \left(n - \frac{d+2h_d}{2d+2h_d}n \right) = \frac{d+2h_d}{2d+2h_d}n.$$

Now we are able to find $|U_k|, |W_j| \geq \frac{d+2h_d}{2d+2h_d}n$, where $k, j \leq L_r/2$. By the inclusion-exclusion principle,

$$|U_k \cap W_j| \geq \frac{h_d}{d+h_d}n.$$

Now each vertex $v \in U_k \cap W_j$ naturally corresponds to a path of length no more than $L_r/2 + L_r/2 = L_r$ with one endpoint in U and the other in W . Notice that there exists $M_{L_r} \in \mathbb{N}$, so that any path of length no more than L_r can intersect at most M_{L_r} other paths of length no more than L_r in G , therefore we can pick at least

$$\lfloor \frac{1}{M_{L_r}} \frac{h_d}{d+h_d}n \rfloor$$

non-intersecting paths which satisfy all requirements stated in the lemma. \square

Proposition 3.5. *Let $a(n) = 2n^{2\gamma} + L_{\delta_0}$, where δ_0 is as in Lemma 3.3 and L_{δ_0} is as in Lemma 3.4. Then for asymptotically almost every $G \sim \mathcal{G}(n, d)$,*

$$\sup_{U \subset \mathcal{V}_G} \mathbb{P}_G\{\xi_{a(n)}^U \neq \emptyset, \xi_{a(n)}^U \neq \xi_{a(n)}^{\mathcal{V}_G}\} = o(1).$$

Proof. It suffices to show

$$\sup_{v \in \mathcal{V}_G} \mathbb{P}_G\{\xi_{a(n)}^v \neq \emptyset, \xi_{a(n)}^v \neq \xi_{a(n)}^{\mathcal{V}_G}\} = o(n^{-1}), \quad (1)$$

because for arbitrary $U \subset \mathcal{V}_G$,

$$\mathbb{P}_G\{\xi_{a(n)}^U \neq \emptyset, \xi_{a(n)}^U \neq \xi_{a(n)}^{\mathcal{V}_G}\} \leq \sum_{v \in U} \mathbb{P}_G\{\xi_{a(n)}^v \neq \emptyset, \xi_{a(n)}^v \neq \xi_{a(n)}^{\mathcal{V}_G}\}.$$

Moreover, in order to show (1) it suffices to show the following inequality by a union bound,

$$\sup_{v \in \mathcal{V}_G, u \in \mathcal{V}_G} \mathbb{P}_G\{\xi_{a(n)}^v \neq \emptyset, u \notin \xi_{a(n)}^v, u \in \xi_{a(n)}^{\mathcal{V}_G}\} = o(n^{-2}). \quad (2)$$

Recall the graphical representation and the dual contact process in [9]. To show (2), we chop the time interval $[0, a(n)]$ into $[0, n^{2\gamma} + L_{\delta_0}]$ and $[n^{2\gamma} + L_{\delta_0}, a(n)]$. We run ξ_t^v for time $n^{2\gamma} + L_{\delta_0}$ in the first subinterval and run the dual process $\hat{\xi}_t^u$ for time $n^{2\gamma}$ in the second subinterval, where time t for the dual process corresponds to time $a(n) - t$ for the original process. In particular, time 0 for the dual process corresponds to time $a(n)$ for the original one, and time $n^{2\gamma}$ for the dual process corresponds to time $n^{2\gamma} + L_{\delta_0}$ for the original one.

Notice that ξ_t^v and $\hat{\xi}_t^u$ are two independent processes. Then we have the following upper bound of (2),

$$\begin{aligned} & \mathbb{P}_G\{\xi_{a(n)}^v \neq \emptyset, u \notin \xi_{a(n)}^v, u \in \xi_{a(n)}^{\mathcal{V}_G}\} \leq \mathbb{P}_G\{\xi_{n^{2\gamma}}^v \neq \emptyset, \hat{\xi}_{n^{2\gamma}}^u \neq \emptyset, \xi_{n^{2\gamma}+L_{\delta_0}}^v \cap \hat{\xi}_{n^{2\gamma}}^u = \emptyset\} \\ & \leq \mathbb{P}_G\{\xi_{n^{2\gamma}+L_{\delta_0}}^v \cap \hat{\xi}_{n^{2\gamma}}^u = \emptyset \mid \xi_{n^{2\gamma}}^v \neq \emptyset, \hat{\xi}_{n^{2\gamma}}^u \neq \emptyset\}. \end{aligned}$$

Our goal is to show the following bound holds uniformly for u, v :

$$\mathbb{P}_G\{\xi_{n^{2\gamma}+L_{\delta_0}}^v \cap \hat{\xi}_{n^{2\gamma}}^u = \emptyset \mid \xi_{n^{2\gamma}}^v \neq \emptyset, \hat{\xi}_{n^{2\gamma}}^u \neq \emptyset\} = o(n^{-2}). \quad (3)$$

By Lemma 3.3,

$$\mathbb{P}_G\{|\xi_{n^{2\gamma}}^v| \geq \delta_0 n, |\hat{\xi}_{n^{2\gamma}}^u| \geq \delta_0 n \mid \xi_{n^{2\gamma}}^v \neq \emptyset, \hat{\xi}_{n^{2\gamma}}^u \neq \emptyset\} = (1 - o(n^{-2}))^2 = 1 - o(n^{-2}). \quad (4)$$

Given $\{|\xi_{n^{2\gamma}}^v| \geq \delta_0 n, |\hat{\xi}_{n^{2\gamma}}^u| \geq \delta_0 n\}$, by Lemma 3.4, there will be $c_{\delta_0} n$ non-intersecting paths of length no more than L_{δ_0} with one endpoint in $\xi_{n^{2\gamma}}^v$ and the other in $\hat{\xi}_{n^{2\gamma}}^u$. We call it a *success* on a path v_0, v_1, \dots, v_k with endpoints $v_0 \in \xi_{n^{2\gamma}}^v$ and $v_k \in \hat{\xi}_{n^{2\gamma}}^u$, if in time L_{δ_0} , the infection at v_0 spreads to v_k within this path. For non-intersecting paths, successes on paths are independent events. Also, using similar argument as in proof of Lemma 3.3, it is easy to see that there exists $p_{\delta_0} > 0$ so that the probability of a success is at least p_{δ_0} on a path of length no more than L_{δ_0} .

In order to observe $\{\xi_{n^{2\gamma}+L_{\delta_0}}^v \cap \hat{\xi}_{n^{2\gamma}}^u \neq \emptyset\}$, it suffices that on one of these $c_{\delta_0} n$ paths we observe a success, each having probability at least $p_{\delta_0} > 0$. Therefore the chance of observing at least 1 success is at least $1 - o(n^{-2})$ by an easy binomial calculation.

Combining all arguments above, we see

$$\begin{aligned} & \mathbb{P}_G\{\xi_{n^{2\gamma}+L_{\delta_0}}^v \cap \hat{\xi}_{n^{2\gamma}}^u \neq \emptyset \mid \xi_{n^{2\gamma}}^v \neq \emptyset, \hat{\xi}_{n^{2\gamma}}^u \neq \emptyset\} \\ & \geq \mathbb{P}_G\{|\xi_{n^{2\gamma}}^v| \geq \delta_0 n, |\hat{\xi}_{n^{2\gamma}}^u| \geq \delta_0 n \mid \xi_{n^{2\gamma}}^v \neq \emptyset, \hat{\xi}_{n^{2\gamma}}^u \neq \emptyset\} \times (1 - o(n^{-2})) \\ & = (1 - o(n^{-2}))^2 \\ & = 1 - o(n^{-2}), \end{aligned}$$

which finishes our proof. \square

Proof of Theorem 1.2. By Proposition 3.1, take $b(n) = \exp(\beta n)$ as in Theorem 1.1 and $a(n) = 2n^{2\gamma} + L_{\delta_0}$ as in Proposition 3.5. \square

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References

- [1] Béla Bollobás. The isoperimetric number of random regular graphs. *European J. Combin.*, 9(3):241–244, 1988.
- [2] Béla Bollobás and W. Fernandez de la Vega. The diameter of random regular graphs. *Combinatorica*, 2(2):125–134, 1982.
- [3] Marzio Cassandro, Antonio Galves, Enzo Olivieri, and Maria Eulália Vares. Metastable behavior of stochastic dynamics: a pathwise approach. *J. Statist. Phys.*, 35(5-6):603–634, 1984.
- [4] Michael Cranston, Thomas Mountford, Jean-Christophe Mourrat, and Daniel Valesin. The contact process on finite homogeneous trees revisited. *ALEA Lat. Am. J. Probab. Math. Stat.*, 11(2):385–408, 2014.

- [5] Shlomo Hoory, Nathan Linial, and Avi Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561 (electronic), 2006.
- [6] Brett Kolesnik and Nick Wormald. Lower bounds for the isoperimetric numbers of random regular graphs. *SIAM J. Discrete Math.*, 28(1):553–575, 2014.
- [7] Steven Lalley and Wei Su. Contact process on random regular graphs. *arXiv:1502.07421*.
- [8] Thomas Liggett. Multiple transition points for the contact process on the binary tree. *Ann. Probab.*, 24(4):1675–1710, 1996.
- [9] Thomas M. Liggett. *Interacting particle systems*. Classics in Mathematics. Springer-Verlag, Berlin, 2005. Reprint of the 1985 original.
- [10] Eyal Lubetzky and Allan Sly. Cutoff phenomena for random walks on random regular graphs. *Duke Math. J.*, 153(3):475–510, 2010.
- [11] Neal Madras and Rinaldo Schinazi. Branching random walks on trees. *Stochastic Process. Appl.*, 42(2):255–267, 1992.
- [12] Gregory Morrow, Rinaldo Schinazi, and Yu Zhang. The critical contact process on a homogeneous tree. *J. Appl. Probab.*, 31(1):250–255, 1994.
- [13] Thomas Mountford. A metastable result for the finite multidimensional contact process. *Canad. Math. Bull.*, 36(2):216–226, 1993.
- [14] Thomas Mountford, Jean-Christophe Mourrat, Daniel Valesin, and Qiang Yao. Exponential extinction time of the contact process on finite graphs. *arXiv:1203.2972*.
- [15] Thomas Mountford, Daniel Valesin, and Qiang Yao. Metastable densities for the contact process on power law random graphs. *Electron. J. Probab.*, 18:No. 103, 36, 2013.
- [16] Jean-Christophe Mourrat and Daniel Valesin. Phase transition of the contact process on random regular graphs. *arXiv:1405.0865*.
- [17] Robin Pemantle. The contact process on trees. *Ann. Probab.*, 20(4):2089–2116, 1992.
- [18] Roberto H. Schonmann. Metastability for the contact process. *J. Statist. Phys.*, 41(3-4):445–464, 1985.
- [19] Alan Stacey. The existence of an intermediate phase for the contact process on trees. *Ann. Probab.*, 24(4):1711–1726, 1996.